

DOUBLE COMPLEXES AND VANISHING OF NOVIKOV COHOMOLOGY

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ABSTRACT. We consider non-standard totalisation functors for double complexes, involving left or right truncated products. We show how properties of these imply that the algebraic mapping torus of a self map h of a cochain complex of finitely presented modules has trivial negative NOVIKOV cohomology, and has trivial positive NOVIKOV cohomology provided h is a quasi-isomorphism. As an application we obtain a new and transparent proof that a finitely dominated cochain complex over a LAURENT polynomial ring has trivial (positive and negative) NOVIKOV cohomology.

Finiteness conditions for chain complexes of modules play an important role in both algebra and topology. For example, given a group G one might ask whether the trivial G -module \mathbb{Z} admits a resolution by finitely generated projective $\mathbb{Z}[G]$ -modules; existence of such resolutions is relevant for the study of group cohomology of G , and has applications in the theory of duality groups [B75]. For topologists, finite domination of chain complexes is related, among other things, to questions about finiteness of CW complexes, the topology of ends of manifolds, and obstructions for the existence of non-singular closed 1-forms [Ran95, S06].

A cochain complex C of $R[z, z^{-1}]$ -modules is called *finitely dominated* if it is homotopy equivalent, as a complex of R -modules, to a bounded complex of finitely generated projective R -modules. Finite domination of C can be characterised in various ways; BROWN considered compatibility of the functors $M \mapsto H_*(C; M)$ and $M \mapsto H^*(C; M)$ with products and direct limits, respectively [B75, Theorem 1], while RANICKI showed that C is finitely dominated if and only if the NOVIKOV cohomology of C is trivial [Ran95, Theorem 2] (see also Definition 2.3 and Corollary 2.7 below).

Our approach to NOVIKOV cohomology is elementary, and involves a non-standard totalisation functor for double complexes. Rewriting mapping tori as total complexes of suitable double complexes, cf. Remark 2.8 below, we prove a vanishing result for NOVIKOV cohomology (Theorem 2.5). As an application we obtain a new proof of RANICKI's necessary criterion for finite domination over LAURENT polynomial rings in one variable (Corollary 2.7). — The case of several indeterminates is discussed in papers by SCHÜTZ [S06], and by HÜTTEMANN and QUINN [HQ11].

Date: November 29, 2011.

2000 *Mathematics Subject Classification.* Primary 18G35; Secondary 55U15.

This work was supported by the Engineering and Physical Sciences Research Council [grant number EP/H018743/1].

1. TRUNCATED PRODUCT TOTALISATION OF DOUBLE COMPLEXES

Let R be a ring with unit. A *double complex* $D^{*,*}$ is a $\mathbb{Z} \times \mathbb{Z}$ -indexed collection $(D^{p,q})_{p,q \in \mathbb{Z}}$ of right R -modules together with “horizontal” and “vertical” differentials

$$d^h: D^{p,q} \longrightarrow D^{p+1,q} \quad \text{and} \quad d^v: D^{p,q} \longrightarrow D^{p,q+1}$$

which satisfy the conditions

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad d^h \circ d^v = -d^v \circ d^h.$$

Note that the differentials anti-commute. A “horizontal” cochain complex in the category of “vertical” cochain complexes of right R -modules can be converted to a double complex in this sense by changing the differential of the p th column by the sign $(-1)^p$. — We will in general consider unbounded double complexes so that $D^{p,q} \neq 0$ may occur for $|p|$ and $|q|$ arbitrarily large.

There are two standard ways to convert a double complex into a cochain complex via “totalisation”, one involving direct sums, and one involving direct products. The former results in a cochain complex $\text{Tot}_{\oplus} D^{*,*}$ given by

$$(\text{Tot}_{\oplus} D^{*,*})^n = \bigoplus_{p \in \mathbb{Z}} D^{p, n-p}$$

with coboundary $d = d^h + d^v$, the latter is defined analogously with “ \oplus ” above replaced by “ \prod ”.

In this paper, which was partially inspired by a preprint of BERGMAN [B11, §6], we will consider two non-standard totalisation functors formed by using truncated products. Given a \mathbb{Z} -indexed family of modules M_i we define the *left truncated product* to be the module

$$\prod^{\text{lt}}_i M_i = \bigoplus_{i < 0} M_i \oplus \prod_{i \geq 0} M_i;$$

the elements of this truncated product are “sequences” $(m_i)_{i \in \mathbb{Z}}$ with $m_i \in M_i$ such that $m_i = 0$ for $i \ll 0$, which we might also write in the form $(m_i)_{i \geq k}$ or even $\sum_{i \geq k} m_i z^i$ with z being an indeterminate. The latter notation suggests thinking of such a sequence as a formal LAURENT series with coefficients in the modules M_i . For emphasis and ease of notation we introduce special notation for the case that all the M_i are the same module M ; we let $M((z))$ denote the module of formal LAURENT series with coefficients in M ,

$$M((z)) = \prod^{\text{lt}} M = \left\{ \sum_{i \geq k} m_i z^i \mid k \in \mathbb{Z}, m_i \in M \right\}.$$

Dually we define the *right truncated product* to be the module

$$\prod^{\text{rt}}_i M_i = \prod_{i \leq 0} M_i \oplus \bigoplus_{i > 0} M_i$$

of formal LAURENT series which are finite to the right, and define $M((z^{-1}))$ by setting

$$M((z^{-1})) = \prod^{\text{rt}} M = \left\{ \sum_{i \leq k} m_i z^i \mid k \in \mathbb{Z}, m_i \in M \right\}.$$

Note that $R((z))$ and $R((z^{-1}))$ are rings of formal LAURENT series, also known as NOVIKOV rings; there is a natural identification

$$R((z)) = R[[z]][z^{-1}] \quad \text{and} \quad R((z^{-1})) = R[[z^{-1}]] [z] .$$

The module $M((z))$ has the structure of an $R((z))$ -module given by multiplication of formal LAURENT series. Similarly, $M((z^{-1}))$ can be equipped with an obvious $R((z^{-1}))$ -module structure.

Definition 1.1. Let $D^{*,*}$ be a double complex. We define its *left truncated totalisation* to be the cochain complex ${}^{\text{lt}}\text{Tot } D^{*,*}$ which in cochain level n is given by the left truncated product

$$({}^{\text{lt}}\text{Tot } D^{*,*})^n = \prod_p {}^{\text{lt}} D^{p, n-p} ;$$

the differential is given by $d = d^h + d^v$. — Dually, we define the *right truncated totalisation* to be the cochain complex $\text{Tot}^{\text{rt}} D^{*,*}$ which in chain level n is given by the right truncated product

$$(\text{Tot}^{\text{rt}} D^{*,*})^n = \prod_p {}^{\text{rt}} D^{p, n-p}$$

with differential induced as above.

Proposition 1.2 ([B11, Corollary 29]). *Suppose the double complex $D^{*,*}$ has exact columns. Then ${}^{\text{lt}}\text{Tot } D^{*,*}$ is acyclic. Dually, if $D^{*,*}$ has exact rows then $\text{Tot}^{\text{rt}} D^{*,*}$ is acyclic.*

Proof. We prove the first statement only. Abbreviate ${}^{\text{lt}}\text{Tot } D^{*,*}$ by C . Suppose $x \in C^n$ is a cocycle. We can write $x = (x_i)_{i \geq k}$ with $x_i \in D^{i, n-i}$, and setting $x_j = 0$ for $j < k$ the condition $d(x) = 0$ translates into

$$d^v(x_i) + d^h(x_{i-1}) = 0 \quad \text{for } i \geq k . \quad (1)$$

Set $y_j = 0$ for $j < k$. Suppose by induction on i , starting with $i = k$, that we have constructed $y_j \in D^{j, n-j-1}$ for $j < i$ such that

$$d^v(y_{j-1}) + d^h(y_{j-2}) = x_{j-1} \quad \text{for } j \leq i . \quad (2)$$

This implies that

$$\begin{aligned} d^v(x_i - d^h(y_{i-1})) &= (d^v(x_i) - d^v d^h(y_{i-1})) \\ &= (d^v(x_i) + d^h d^v(y_{i-1})) \\ &= (d^v(x_i) + d^h(x_{i-1} - d^h(y_{i-2}))) && \text{(by (2))} \\ &= d^v(x_i) + d^h(x_{i-1}) \\ &= 0 && \text{(by (1))} \end{aligned}$$

so that, by exactness of columns, there exists $y_i \in D^{i, n-i-1}$ with $d^v(y_i) = (x_i - d^h(y_{i-1}))$ or, equivalently, $d^v(y_i) + d^h(y_{i-1}) = x_i$.

This completes the inductive construction. It remains to observe that relation (2) is now satisfied for all $j \in \mathbb{Z}$ which precisely means that the element $(y_i)_{i \geq k} \in C^{n-1}$ is mapped to x under the coboundary map of C . Consequently, x represents the trivial cohomology class in $H^n(C)$ so that $H^n(C) = 0$. \square

Remark 1.3. The Proposition does not hold for the totalisation functor Tot_\oplus in place of ${}^{\text{lt}}\text{Tot}$ or Tot^{rt} . For example, let $D^{*,*}$ be the double complex defined by setting $D^{p,-p} = D^{p,-p-1} = \mathbb{Z}$ and all other entries 0; the horizontal and vertical differentials are given by $-\text{id}_\mathbb{Z}$ and multiplication by 2 where possible, respectively. This double complex has exact rows, but the element $1 \in D^{0,0} \subset (\text{Tot}_\oplus D^{*,*})^0$ is a cocycle representing a non-zero cohomology class in $H^0 \text{Tot}_\oplus(D^{*,*})$. The same element represents a non-trivial cohomology class in $H^0 {}^{\text{lt}}\text{Tot}(D^{*,*})$ as well.

2. NOVIKOV COHOMOLOGY OF ALGEBRAIC MAPPING TORI

Lemma 2.1. *Suppose that M is a finitely presented right R -module. There is a natural $R((z))$ -linear isomorphism*

$$\Phi_M: M \otimes_R R((z)) \xrightarrow{\cong} M((z)), \quad m \otimes \sum_{i \geq k} r_i z^i \mapsto \sum_{i \geq k} m r_i z^i,$$

and a similar isomorphism $\Psi_M: M \otimes_R R((z^{-1})) \xrightarrow{\cong} M((z^{-1}))$.

Proof. We give the proof for Φ_M only. First suppose that F is a free module with basis e_1, e_2, \dots, e_t . Then every $x \in F \otimes_R R((z))$ can be written uniquely in the form $x = \sum_{j=1}^t e_j \otimes f_j$ with $f_j \in R((z))$. There exist elements $k \in \mathbb{Z}$ and $r_{ij} \in R$ with $f_j = \sum_{i \geq k} r_{ij} z^i$, and Φ_F is given by setting

$$\Phi_F(x) = \sum_{i \geq k} \left(\sum_{j=1}^t e_j r_{ij} \right) z^i.$$

This is a well-defined R -module homomorphism. It is injective since the e_j form a basis of F ; in detail, $\Phi_F(x) = 0$ means that $\sum_j r_{ij} e_j = 0$ for all i so that, by linear independence of the e_j , we have $r_{ij} = 0$ for all i and j . But this means $x = 0$. — To prove surjectivity, let $g = \sum_{i \geq k} m_i z^i \in F((z))$ be given. Since the e_j generate F there are elements $r_{ij} \in R$ with $m_i = \sum_j e_j r_{ij}$. Set $f_j = \sum_{i \geq k} r_{ij} z^i$ and $x = \sum_j (e_j \otimes f_j)$. Then $\Phi_F(x) = g$, by construction.

For the general case, choose a presentation $G \longrightarrow F \longrightarrow M \longrightarrow 0$ of M , with F and G both finitely generated free. The functor $N \mapsto N((z))$ is certainly exact (for a map f we let $f((z))$ denote the map f applied componentwise), so we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} G \otimes_R R((z)) & \longrightarrow & F \otimes_R R((z)) & \longrightarrow & M \otimes_R R((z)) & \longrightarrow & 0 \\ \Phi_G \downarrow \cong & & \Phi_F \downarrow \cong & & \Phi_M \downarrow \text{!} & & \\ 0 & \longrightarrow & G((z)) & \longrightarrow & F((z)) & \longrightarrow & M((z)) \longrightarrow 0 \end{array} \quad (3)$$

where the dashed arrow is Φ_M . In fact, every element $x \in M \otimes_R R((z))$ can be written (in at least one way) in the form $x = \sum_{j=1}^s m_j \otimes f_j$, with $m_j \in M$ and $f_j = \sum_{i \geq k} r_{ij} z^i \in R((z))$, and $\Phi_M(x) = \sum_{i \geq k} \left(\sum_{j=1}^s m_j r_{ij} \right) z^i$; commutativity of (3) shows that this is well defined. By the Five Lemma, the map Φ_M is an isomorphism in general as claimed. \square

Remark 2.2. The lemma fails for modules which are not finitely generated. Specifically, if M is free of infinite rank one can still define a map $M \otimes_R R((z)) \longrightarrow M((z))$, essentially in the same way as above, and linear independence of basis elements guarantees that this map is injective. Its image consists precisely of those formal LAURENT series $\sum_{i \geq k} m_i z^i$ which have the property that the submodule of M generated by the set of coefficients $\{m_i \mid i \geq k\}$ is finitely generated. Using this and a diagram chase in (3) one can show that Φ_M is surjective whenever M is finitely generated; in that case Φ_M is injective as well if and only if M is finitely presented.

Definition 2.3. Let B be a cochain complex of $R[z, z^{-1}]$ -modules. The *positive NOVIKOV cohomology* is the cohomology of the cochain complex $B \otimes_{R[z, z^{-1}]} R((z))$. The *negative NOVIKOV cohomology* is the cohomology of the cochain complex $B \otimes_{R[z, z^{-1}]} R((z^{-1}))$.

Definition 2.4. Let C be a cochain complex of right R -modules, and let $h: C \longrightarrow C$ be a cochain map. The *mapping torus* $T(h)$ of h is defined by

$$T(h) = \text{Cone} \left(C \otimes_R R[z, z^{-1}] \xrightarrow{h \otimes \text{id} - \text{id} \otimes z} C \otimes_R R[z, z^{-1}] \right)$$

where the map “ z ” denotes the self map of $R[z, z^{-1}]$ given by multiplication by the indeterminate z .

In this definition “Cone” stands for the algebraic mapping cone; if a map of cochain complexes $f: X \longrightarrow Y$ is considered as a double complex $D^{*,*}$ concentrated in columns $p = -1, 0$ with horizontal differential f , and differential of X changed by a sign -1 , then $\text{Cone}(f) = \text{Tot}_{\oplus} D^{*,*}$. Explicitly, we have $\text{Cone}(f)^n = X^{n+1} \oplus Y^n$, and the differential is given by the following formula:

$$\begin{aligned} \text{Cone}(f)^n &= X^{n+1} \oplus Y^n \longrightarrow X^{n+2} \oplus Y^{n+1} = \text{Cone}(f)^{n+1} \\ (x, y) &\mapsto (-d(x), f(x) + d(y)) \end{aligned}$$

Theorem 2.5. *Let C be a (possibly unbounded) cochain complex of finitely presented right R -modules, and let $h: C \longrightarrow C$ be an arbitrary cochain map. Then the negative NOVIKOV cohomology of the mapping torus $T(h)$ of h is trivial, i.e., the cochain complex $T(h) \otimes_{R[z, z^{-1}]} R((z^{-1}))$ is acyclic. — If h is a quasi-isomorphism, then the positive NOVIKOV homology of $T(h)$ is trivial as well, i.e., the cochain complex $T(h) \otimes_{R[z, z^{-1}]} R((z))$ is acyclic in this case.*

Proof. We deal with negative NOVIKOV cohomology first. Since tensor products are additive, we have an equality of cochain complexes

$$T(h) \otimes_{R[z, z^{-1}]} R((z^{-1})) = \text{Cone} \left(C \otimes_R R((z^{-1})) \xrightarrow{h \otimes \text{id} - \text{id} \otimes z} C \otimes_R R((z^{-1})) \right).$$

Using Lemma 2.1 we identify the complex $C \otimes_R R((z^{-1}))$ with $C((z^{-1}))$. We can now write

$$T(h) \otimes_{R[z, z^{-1}]} R((z^{-1})) = \text{Tot}^{\text{rt}}(D^{*,*}) \quad (4)$$

where $D^{*,*}$ is defined as follows:

$$\begin{aligned} D^{p,q} &= C^{p+q+1} \oplus C^{p+q} \\ d^h: D^{p,q} &\longrightarrow D^{p+1,q} \\ (x, y) &\mapsto (0, -x) \\ d^v: D^{p,q} &\longrightarrow D^{p,q+1} \\ (x, y) &\mapsto (-d^C(x), h(x) + d^C(y)) \end{aligned}$$

Here d^C denotes the coboundary map in the complex C . We have $d^h \circ d^h = 0$, and the p th column $D^{p,*}$ of $D^{*,*}$ is the p th shift of $\text{Cone}(h)$ so that $d^v \circ d^v = 0$. Finally, the differentials anti-commute: for a typical element $(x, y) \in D^{p,q} = C^{p+q+1} \oplus C^{p+q}$ we have

$$\begin{aligned} d^v \circ d^h(x, y) &= d^v(0, -x) = (0, d^C(-x)) = -(0, d^C(x)) \\ &= -d^h((-d^C(x), h(x) + d^C(y))) = -d^h \circ d^v(x, y). \end{aligned}$$

To complete the identification given in (4) we note that the p th column of $D^{*,*}$ corresponds to the terms with coefficient z^p in the formal LAURENT series notation. — Now the rows of $D^{*,*}$ are clearly exact so that $\text{Tot}^{\text{rt}}(D^{*,*})$ is acyclic by Proposition 1.2.

For positive NOVIKOV cohomology we note that since $C \otimes_R R((z)) = C((z))$ we can identify $T(h) \otimes_{R[z, z^{-1}]} R((z))$ with ${}^{\text{lt}}\text{Tot}(D^{*,*})$. If h is a quasi-isomorphism then $\text{Cone}(h)$ is acyclic so that the columns of $D^{*,*}$ are exact. By Proposition 1.2, ${}^{\text{lt}}\text{Tot}(D^{*,*})$ is acyclic. \square

Remark 2.6. Suppose h is the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by multiplication by 2, considered as a cochain map concentrated in cochain degree 0. Then the $D^{*,*}$ in the proof above is the double complex described in Remark 1.3 which has the property that ${}^{\text{lt}}\text{Tot}(D^{*,*})$ is not acyclic. This provides an example of a map h whose mapping torus has trivial negative but non-trivial positive NOVIKOV cohomology.

The asymmetry stems from the fact that the definition of mapping tori involves a choice. One could have defined the mapping torus of h as the mapping cone of $h \otimes \text{id} - \text{id} \otimes z^{-1}$ in which case the roles of positive and negative NOVIKOV cohomology in Theorem 2.5 are reversed. This can be shown by identifying the p th column of $D^{*,*}$ in the proof above with the coefficients of z^{-p} in the LAURENT series notation, or by using double complexes with differentials going down and left (in which case the roles of ${}^{\text{lt}}\text{Tot}$ and Tot^{rt} are swapped in Proposition 1.2).

Corollary 2.7. *Suppose that C is a bounded above cochain complex of projective right $R[z, z^{-1}]$ -modules. Suppose further that C is homotopy equivalent, as an R -module complex, to a bounded complex B of finitely generated projective right R -modules. Then C has trivial positive and negative NOVIKOV cohomology, that is, the two cochain complexes $C \otimes_{R[z, z^{-1}]} R((z))$ and $C \otimes_{R[z, z^{-1}]} R((z^{-1}))$ are acyclic.*

Proof. Let $f: C \longrightarrow B$ and $g: B \longrightarrow C$ mutually inverse R -linear homotopy equivalences. There are $R[z, z^{-1}]$ -linear homotopy equivalences

$$C \longleftarrow T(zgf) \longrightarrow T(fzg)$$

where “ z ” denotes the self map given by multiplication by z ; a proof can be found, for example, in [HQ11, §§2–3]. It follows that the NOVIKOV cohomology of C and of $T(fzg)$ are the same. Now fzg is a homotopy equivalence as z acts invertibly on C , and Theorem 2.5 assures us that $T(fzg)$ has trivial positive and negative NOVIKOV cohomology. \square

This Corollary is the “only-if” part of a result obtained by RANICKI [Ran95, Theorem 2] using different methods; the present proof has the advantage of being completely elementary.

Remark 2.8. With $D^{*,*}$ as in the proof of Theorem 2.5 we can identify $T(h)$ with $\text{Tot}_{\oplus} D^{*,*}$. Note that $D^{*,*}$ has exact rows, and has exact columns if h is a quasi-isomorphism. In view of Remark 1.3 this does *not* imply that $T(h)$ is acyclic.

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